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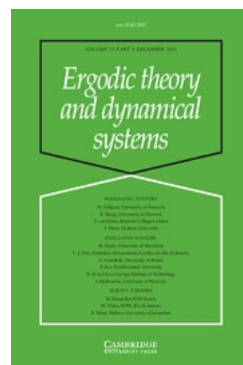
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# Parameter rigid actions of simply connected nilpotent Lie groups

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**Abstract.** We show that for a locally free  $C^\infty$ -action of a connected and simply connected nilpotent Lie group on a compact manifold, if every real-valued cocycle is cohomologous to a constant cocycle, then the action is parameter rigid. The converse is true if the action has a dense orbit. Using this, we construct parameter rigid actions of simply connected nilpotent Lie groups whose Lie algebras admit rational structures with graduations. This generalizes the results of dos Santos [Parameter rigid actions of the Heisenberg groups. *Ergod. Th. & Dynam. Sys.* **27** (2007), 1719–1735] concerning the Heisenberg groups.

## 1. Introduction

Let  $G$  be a connected Lie group with Lie algebra  $\mathfrak{g}$  and  $M$  a  $C^\infty$ -manifold without boundary. Let  $\rho : M \times G \rightarrow M$  be a  $C^\infty$  right action. We call  $\rho$  *locally free* if every isotropy subgroup of  $\rho$  is discrete in  $G$ . Assume that  $\rho$  is locally free. Then we have the orbit foliation  $\mathcal{F}$  of  $\rho$  whose tangent bundle  $T\mathcal{F}$  is naturally isomorphic to a trivial bundle  $M \times \mathfrak{g}$ .

The action  $\rho$  is *parameter rigid* if any action  $\rho'$  of  $G$  on  $M$  with the same orbit foliation  $\mathcal{F}$  is  $C^\infty$ -conjugate to  $\rho$ ; more precisely, there exist an automorphism  $\Phi$  of  $G$  and a  $C^\infty$ -diffeomorphism  $F$  of  $M$  which preserves each leaf of  $\mathcal{F}$  and is homotopic to the identity through  $C^\infty$ -maps preserving each leaf of  $\mathcal{F}$  such that

$$F(\rho(x, g)) = \rho'(F(x), \Phi(g))$$

for all  $x \in M$  and  $g \in G$ .

Parameter rigidity of actions has been studied by several authors, for instance, Katok and Spatzier [3], Matsumoto and Mitsumatsu [4], Mieczkowski [5], Ramírez [7] and dos Santos [8]. Most of the known examples of parameter rigid actions are those of abelian groups, and non-abelian actions have not been considered so much.

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Parameter rigidity is closely related to cocycles over actions. Now suppose  $G$  is contractible and  $M$  is compact. Let  $H$  be a Lie group. A  $C^\infty$ -map  $c : M \times G \rightarrow H$  is called a  $H$ -valued cocycle over  $\rho$  if  $c$  satisfies

$$c(x, gg') = c(x, g)c(\rho(x, g), g')$$

for all  $x \in M$  and  $g, g' \in G$ .

A cocycle  $c$  is constant if  $c(x, g)$  is independent of  $x$ . A constant cocycle is just a homomorphism  $G \rightarrow H$ .

$H$ -valued cocycles  $c, c'$  are cohomologous if there exists a  $C^\infty$ -map  $P : M \rightarrow H$  such that

$$c(x, g) = P(x)^{-1}c'(x, g)P(\rho(x, g))$$

for all  $x \in M$  and  $g \in G$ .

The action  $\rho$  is  $H$ -valued cocycle rigid if every  $H$ -valued cocycle over  $\rho$  is cohomologous to a constant cocycle.

PROPOSITION 1. [4] If  $\rho$  is  $G$ -valued cocycle rigid, then it is parameter rigid.

*Remark.* In [4] Matsumoto and Mitsumatsu assume that  $\rho$  has at least one trivial isotropy subgroup, but this assumption is not necessary.

PROPOSITION 2. [4] When  $G = \mathbb{R}^n$ , the following are equivalent:

- (1)  $\rho$  is  $\mathbb{R}$ -valued cocycle rigid;
- (2)  $\rho$  is  $\mathbb{R}^n$ -valued cocycle rigid;
- (3)  $\rho$  is parameter rigid.

*Remark.* The equivalence of the first two conditions is obvious.

In this paper we consider actions of simply connected nilpotent Lie groups. In [8], dos Santos proved that, for actions of a Heisenberg group  $H_n$ ,  $\mathbb{R}$ -valued cocycle rigidity implies  $H_n$ -valued cocycle rigidity, and, using this, he constructed parameter rigid actions of Heisenberg groups. To the best of my knowledge these are the only known non-trivial parameter rigid actions of non-abelian nilpotent Lie groups. We prove the following.

THEOREM 1. Let  $N$  be a connected and simply connected nilpotent Lie group,  $M$  a compact manifold and  $\rho$  a locally free  $C^\infty$ -action of  $N$  on  $M$ . Then, the following are equivalent:

- (1)  $\rho$  is  $\mathbb{R}$ -valued cocycle rigid;
- (2)  $\rho$  is  $N$ -valued cocycle rigid;
- (3)  $\rho$  is parameter rigid and every orbitwise constant real-valued  $C^\infty$ -function of  $\rho$  on  $M$  is constant on  $M$ .

This theorem enables us to construct parameter rigid actions of nilpotent Lie groups. The most interesting one is the following.

THEOREM 2. [7] Let  $N$  denote the group of all upper triangular real matrices with 1 on the diagonal,  $\Gamma$  a cocompact lattice of  $\mathrm{SL}(n, \mathbb{R})$  and  $\rho$  the action of  $N$  on  $\Gamma \backslash \mathrm{SL}(n, \mathbb{R})$  by right multiplication. If  $n \geq 4$ ,  $\rho$  is  $\mathbb{R}$ -valued cocycle rigid.

*Remark.* In [7], Ramírez proved more general theorems.

**COROLLARY.** *The above action  $\rho$  is parameter rigid.*

In §4 we construct parameter rigid actions of nilpotent groups using Theorem 1. It is a generalization of dos Santos' example. Let  $N$  be a connected and simply connected nilpotent Lie group and  $\Gamma, \Lambda$  be lattices in  $N$ . Consider the action of  $\Lambda$  on  $\Gamma \backslash N$  by right multiplication and let  $\tilde{\rho}$  be its suspended action of  $N$ .

**THEOREM 3.** *If  $\Lambda$  is Diophantine with respect to  $\Gamma$ , then the action  $\tilde{\rho}$  of  $N$  is parameter rigid.*

For the definition of Diophantine lattices, see §4.

## 2. Preliminaries

Let  $G$  be a contractible Lie group with Lie algebra  $\mathfrak{g}$ ,  $M$  a compact manifold and  $\rho$  a locally free action of  $G$  on  $M$  with orbit foliation  $\mathcal{F}$ . Let  $H$  be a Lie group with Lie algebra  $\mathfrak{h}$ . We denote by  $\Omega^p(\mathcal{F}, \mathfrak{h})$  the set of all  $C^\infty$ -sections of  $\text{Hom}(\bigwedge^p T\mathcal{F}, \mathfrak{h})$ . The exterior derivative

$$d_{\mathcal{F}} : \Omega^p(\mathcal{F}, \mathfrak{h}) \rightarrow \Omega^{p+1}(\mathcal{F}, \mathfrak{h})$$

is defined since  $T\mathcal{F}$  is integrable.

By differentiating,  $H$ -valued cocycles over  $\rho$  are in one-to-one correspondence with  $\mathfrak{h}$ -valued leafwise one-forms  $\omega \in \Omega^1(\mathcal{F}, \mathfrak{h})$  such that

$$d_{\mathcal{F}}\omega + [\omega, \omega] = 0.$$

**PROPOSITION 3.** *Let  $c_1, c_2$  be  $H$ -valued cocycles over  $\rho$  and let  $\omega_1, \omega_2$  be corresponding differential forms. For a  $C^\infty$ -map  $P : M \rightarrow H$ , the following are equivalent:*

- (1)  $c_1(x, g) = P(x)^{-1}c_2(x, g)P(\rho(x, g))$  for all  $x \in M$  and  $g \in G$ ;
- (2)  $\omega_1 = \text{Ad}(P^{-1})\omega_2 + P^*\theta$  where  $\theta \in \Omega^1(H, \mathfrak{h})$  is the left Maurer–Cartan form on  $H$ .

**COROLLARY.** [4] *The following are equivalent:*

- (1)  $\rho$  is  $G$ -valued cocycle rigid;
- (2) for each  $\omega \in \Omega^1(\mathcal{F}, \mathfrak{g})$  such that  $d_{\mathcal{F}}\omega + [\omega, \omega] = 0$ , there exist an endomorphism  $\Phi : \mathfrak{g} \rightarrow \mathfrak{g}$  of the Lie algebra and a  $C^\infty$ -map  $P : M \rightarrow G$  such that

$$\omega = \text{Ad}(P^{-1})\Phi + P^*\theta.$$

Proposition 3 is obtained by examining the proof of [4, Corollary 2]. In this paper, we will identify a cocycle with its corresponding differential form.

Let us consider real-valued cocycles. A real-valued cocycle over  $\rho$  is given by  $\omega \in \Omega^1(\mathcal{F}, \mathbb{R})$  satisfying  $d_{\mathcal{F}}\omega = 0$ . Two real-valued cocycles  $\omega_1, \omega_2$  are cohomologous if and only if  $\omega_1 = \omega_2 + d_{\mathcal{F}}P$  for some  $C^\infty$ -function  $P : M \rightarrow \mathbb{R}$ . Leafwise cohomology  $H^*(\mathcal{F})$  of  $\mathcal{F}$  is the cohomology of the cochain complex  $(\Omega^*(\mathcal{F}, \mathbb{R}), d_{\mathcal{F}})$ . Thus  $H^1(\mathcal{F})$  is the set of all equivalence classes of real-valued cocycles.

The identification  $T\mathcal{F} \simeq M \times \mathfrak{g}$  induces a map  $H^*(\mathfrak{g}) \rightarrow H^*(\mathcal{F})$  where  $H^*(\mathfrak{g})$  is the cohomology of the Lie algebra  $\mathfrak{g}$ . By the compactness of  $M$ , this map is injective on  $H^1(\mathfrak{g})$ . Hence we identify  $H^1(\mathfrak{g})$  with its image. Note that  $H^1(\mathfrak{g})$  is the set of all equivalence

classes of constant real-valued cocycles. Thus real-valued cocycle rigidity is equivalent to  $H^1(\mathcal{F}) = H^1(\mathfrak{g})$ .

Notice that  $H^0(\mathcal{F})$  is the set of leafwise constant real-valued  $C^\infty$ -functions of  $\mathcal{F}$  on  $M$  and  $H^0(\mathfrak{g})$  consists of constant functions on  $M$ . Therefore the equivalence of (1) and (3) in Theorem 1 can be stated as follows:  $H^1(\mathcal{F}) = H^1(\mathfrak{n})$  if and only if  $\rho$  is parameter rigid and  $H^0(\mathcal{F}) = H^0(\mathfrak{n})$ .

### 3. Proof of Theorem 1

Let  $N$  be a simply connected nilpotent Lie group with Lie algebra  $\mathfrak{n}$ ,  $M$  a compact manifold and  $\rho$  a locally free action of  $N$  on  $M$  with orbit foliation  $\mathcal{F}$ .

We first prove that  $N$ -valued cocycle rigidity implies real-valued cocycle rigidity. There exist closed subgroups  $N'$  and  $A$  of  $N$  such that  $N' \triangleleft N$ ,  $N = N' \rtimes A$  and  $A \simeq \mathbb{R}$ . Let  $c$  be any real-valued cocycle over  $\rho$ . We regard  $c$  as an  $N$ -valued cocycle over  $\rho$  via the inclusion  $\mathbb{R} \simeq A \hookrightarrow N$ . By  $N$ -valued cocycle rigidity, there exist an endomorphism  $\Phi$  of  $N$  and a  $C^\infty$ -map  $P : M \rightarrow N$  such that  $c(x, g) = P(x)^{-1} \Phi(g) P(\rho(x, g))$  for all  $x \in M$  and  $g \in N$ . Applying the natural projection  $\pi : N \rightarrow A \simeq \mathbb{R}$ , we obtain  $c(x, g) = (\pi \circ P)(x)^{-1} (\pi \circ \Phi)(g) (\pi \circ P)(\rho(x, g))$ . Thus  $c$  is cohomologous to a constant cocycle  $\pi \circ \Phi$ .

Next we assume  $H^1(\mathcal{F}) = H^1(\mathfrak{n})$  and prove  $N$ -valued cocycle rigidity. We need the following two lemmata.

**LEMMA 1.** *Let  $V$  be a finite dimensional real vector space. Assume that  $\omega \in \Omega^1(\mathcal{F}, V)$  satisfies the equation  $d_{\mathcal{F}}\omega = \varphi$ , where  $\varphi \in \text{Hom}(\bigwedge^2 \mathfrak{n}, V)$  is a constant leafwise two-form. Then there exists a constant leafwise one-form  $\psi \in \text{Hom}(\mathfrak{n}, V)$  with  $\varphi = d_{\mathcal{F}}\psi$ .*

*Proof.* Since  $N$  is amenable, there exists an  $N$ -invariant Borel probability measure  $\mu$  on  $M$ . Define  $\psi \in \text{Hom}(\mathfrak{n}, V)$  by

$$\psi(X) = \int_M \omega(X) d\mu,$$

where  $X \in \mathfrak{n}$ . Since  $\varphi(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y])$  for all  $X, Y \in \mathfrak{n}$ , we obtain

$$\varphi(X, Y) = - \int_M \omega([X, Y]) d\mu.$$

Thus

$$d_{\mathcal{F}}\psi(X, Y) = -\psi([X, Y]) = - \int_M \omega([X, Y]) d\mu = \varphi(X, Y),$$

and hence  $d_{\mathcal{F}}\psi = \varphi$ . □

Set  $\mathfrak{n}^1 = \mathfrak{n}$ ,  $\mathfrak{n}^i = [\mathfrak{n}, \mathfrak{n}^{i-1}]$ . Then  $\mathfrak{n}^s \neq 0$ ,  $\mathfrak{n}^{s+1} = 0$  for some  $s$ . For each  $1 \leq i \leq s$ , choose a subspace  $V_i$  with  $\mathfrak{n}^i = V_i \oplus \mathfrak{n}^{i+1}$ , so that  $\mathfrak{n} = \bigoplus_{i=1}^s V_i$ .

**LEMMA 2.** *Let  $\omega \in \Omega^1(\mathcal{F}, \mathfrak{n})$  be such that  $d_{\mathcal{F}}\omega + [\omega, \omega] = 0$ . Decompose  $\omega$  as*

$$\omega = \xi + \omega_k + \omega_{k+1},$$

where  $\xi \in \Omega^1(\mathcal{F}, \bigoplus_{i=1}^{k-1} V_i)$ ,  $\omega_k \in \Omega^1(\mathcal{F}, V_k)$  and  $\omega_{k+1} \in \Omega^1(\mathcal{F}, \mathfrak{n}^{k+1})$ . If  $\xi$  is constant, then there exists  $\omega' \in \Omega^1(\mathcal{F}, \mathfrak{n})$  with  $d_{\mathcal{F}}\omega' + [\omega', \omega'] = 0$  which is cohomologous to  $\omega$  and

such that

$$\omega' = \xi' + \omega'_{k+1},$$

where  $\xi' \in \Omega^1(\mathcal{F}, \bigoplus_{i=1}^k V_i)$  is constant and  $\omega'_{k+1} \in \Omega(\mathcal{F}, \mathfrak{n}^{k+1})$ .

*Proof.* By the cocycle equation,

$$0 = d_{\mathcal{F}}\xi + d_{\mathcal{F}}\omega_k + d_{\mathcal{F}}\omega_{k+1} + [\xi, \xi] + \text{an element of } \Omega^2(\mathcal{F}, \mathfrak{n}^{k+1}).$$

Comparing this with the  $V_k$  component, we see that  $d_{\mathcal{F}}\omega_k$  is constant. Hence, by Lemma 1,  $d_{\mathcal{F}}\omega_k = d_{\mathcal{F}}\psi$  for some  $\psi \in \text{Hom}(\mathfrak{n}, V_k)$ . Since we are assuming that  $H^1(\mathcal{F}) = H^1(\mathfrak{n})$ , there exists  $\psi' \in \text{Hom}(\mathfrak{n}, V_k)$  and a  $C^\infty$ -map  $h : M \rightarrow V_k$  such that

$$\omega_k = \psi + \psi' + d_{\mathcal{F}}h.$$

Put  $P = e^h : M \rightarrow N$ . Let  $x \in M$  and  $X \in T_x\mathcal{F}$ . Choose a path  $x(t)$  such that  $X = (d/dt)x(t)|_{t=0}$ . Let  $\theta \in \Omega^1(N, \mathfrak{n})$  be the left Maurer–Cartan form on  $N$ . Then

$$\begin{aligned} P^*\theta(X) &= \left. \frac{d}{dt} P(x)^{-1} P(x(t)) \right|_{t=0} = \left. \frac{d}{dt} e^{-h(x)} e^{h(x(t))} \right|_{t=0} \\ &= \left. \frac{d}{dt} \exp(-h(x) + h(x(t))) + \text{an element of } \mathfrak{n}^{k+1} \right|_{t=0} \\ &= d_{\mathcal{F}}h(X) + \text{an element of } \mathfrak{n}^{k+1}. \end{aligned}$$

Thus  $P^*\theta = d_{\mathcal{F}}h + \text{an element of } \Omega^1(\mathcal{F}, \mathfrak{n}^{k+1})$ . Note that  $\text{Ad}(P^{-1}) = \exp \text{ad}(-h)$  is the identity on  $\bigoplus_{i=1}^k V_i$  and preserves  $\mathfrak{n}^{k+1}$ . Hence

$$\begin{aligned} \omega - P^*\theta &= \xi + \psi + \psi' + \text{an element of } \Omega^1(\mathcal{F}, \mathfrak{n}^{k+1}) \\ &= \text{Ad}(P^{-1})(\xi + \psi + \psi' + \text{an element of } \Omega^1(\mathcal{F}, \mathfrak{n}^{k+1})). \quad \square \end{aligned}$$

Let  $\omega$  be any  $N$ -valued cocycle. Using Lemma 2, we can exchange  $\omega$  for a cohomologous cocycle whose  $V_1$ -component is constant. Applying Lemma 2 repeatedly, we eventually get a constant cocycle cohomologous to  $\omega$ . This proves  $N$ -valued cocycle rigidity.

Next we assume that  $\rho$  is parameter rigid and  $H^0(\mathcal{F}) = H^0(\mathfrak{n})$ . Let  $\mathfrak{n}^i$  and  $V_i$  be as above. Note that  $\mathfrak{n}^s$  is central in  $\mathfrak{n}$ . Fix a non-zero  $Z \in \mathfrak{n}^s$ .

Let  $[\omega] \in H^1(\mathcal{F})$ . Let  $\omega_0$  be the  $N$ -valued cocycle over  $\rho$  corresponding to the constant cocycle  $\text{id} : N \rightarrow N$ . We call  $\omega_0$  the canonical one-form of  $\rho$ . Fix an  $\epsilon > 0$  and put  $\eta := \omega_0 + \epsilon\omega Z$ . Then  $\eta$  is an  $N$ -valued cocycle over  $\rho$ , since

$$d_{\mathcal{F}}\eta + [\eta, \eta] = d_{\mathcal{F}}\omega_0 + \epsilon(d_{\mathcal{F}}\omega)Z + [\omega_0, \omega_0] = 0.$$

Since  $M$  is compact, we can assume  $\eta_x : T_x\mathcal{F} \rightarrow \mathfrak{n}$  is bijective for all  $x \in M$  by choosing  $\epsilon > 0$  small. There exists a unique action  $\rho'$  of  $N$  on  $M$  whose orbit foliation is  $\mathcal{F}$  and canonical one-form is  $\eta$ . See [1]. By parameter rigidity,  $\rho'$  is conjugate to  $\rho$ . Thus there exist a  $C^\infty$ -map  $P : M \rightarrow N$  and an automorphism  $\Phi$  of  $N$  satisfying

$$\omega_0 + \epsilon\omega Z = \text{Ad}(P^{-1})\Phi_*\omega_0 + P^*\theta. \quad (1)$$

Note that  $\log : N \rightarrow \mathfrak{n}$  is defined since  $N$  is simply connected and nilpotent. Let us decompose  $\omega_0 = \sum_{i=1}^s \omega_{0i}$ ,  $\Phi_*\omega_0 = \sum_{i=1}^s \omega'_{0i}$  and  $\log P = \sum_{i=1}^s P_i$  according to the decomposition  $\mathfrak{n} = \bigoplus_{i=1}^s V_i$ .

LEMMA 3. Assume that  $P_1 = \cdots = P_{k-1} = 0$ ; that is,  $\log P \in \mathfrak{n}^k$ .

- (1) If  $k < s$ , then there exist a  $C^\infty$ -map  $Q : M \rightarrow N$  and an automorphism  $\Psi$  of  $N$  such that

$$\omega_0 + \epsilon \omega Z = \text{Ad}(Q^{-1})\Psi_*\omega_0 + Q^*\theta$$

and  $Q_1 = \cdots = Q_k = 0$ , where  $\log Q = \sum_{i=1}^s Q_i$ .

- (2) If  $k = s$ , then  $\omega$  is cohomologous to a constant cocycle.

*Proof.* For all  $X = (d/dt)x(t)|_{t=0} \in T_x\mathcal{F}$ ,

$$\begin{aligned} P^*\theta(X) &= \frac{d}{dt}P(x)^{-1}P(x(t))\Big|_{t=0} = \frac{d}{dt}\exp\left(-\sum_{i=k}^s P_i(x)\right)\exp\left(\sum_{i=k}^s P_i(x(t))\right)\Big|_{t=0} \\ &= \frac{d}{dt}\exp\left\{\sum_{i=k}^s (P_i(x(t)) - P_i(x)) + \text{an element of } \mathfrak{n}^{k+1}\right\}\Big|_{t=0} \\ &= \frac{d}{dt}\exp(P_k(x(t)) - P_k(x) + \text{an element of } \mathfrak{n}^{k+1})\Big|_{t=0} \\ &= d_{\mathcal{F}}P_k(X) + \text{an element of } \mathfrak{n}^{k+1}. \end{aligned}$$

We have

$$\begin{aligned} \text{Ad}(P^{-1})\Phi_*\omega_0 &= \exp\left(\text{ad}\left(-\sum_{i=k}^s P_i\right)\right)\sum_{i=1}^s \omega'_{0i} \\ &= \sum_{i=1}^s \omega'_{0i} + \text{an element of } \mathfrak{n}^{k+1}. \end{aligned}$$

Comparing this with the  $V_k$ -component of (1) we get

$$\omega_{0k} + \delta_{ks}\epsilon\omega Z = \omega'_{0k} + d_{\mathcal{F}}P_k.$$

When  $k = s$ , the equation

$$\omega Z = \epsilon^{-1}(\omega'_{0s} - \omega_{0s}) + d_{\mathcal{F}}(\epsilon^{-1}P_s)$$

shows that  $\omega$  is cohomologous to a constant cocycle.

If  $k < s$ , then  $d_{\mathcal{F}}P_k = \phi \circ \omega_0$  for some linear map  $\phi : \mathfrak{n} \rightarrow V_k$ . For any  $X \in \mathfrak{n}$ , let  $\tilde{X}$  denote the vector field on  $M$  determined by  $X$  via  $\rho$ . We have  $\tilde{X}P_k = \phi(X)$ , and by integrating over an integral curve  $\gamma$  of  $\tilde{X}$  we get  $P_k(\gamma(T)) - P_k(\gamma(0)) = \phi(X)T$  for all  $T > 0$ . Since  $M$  is compact,  $\phi(X) = 0$ . Therefore  $d_{\mathcal{F}}P_k = 0$ , so  $P_k$  is constant on each leaf of  $\mathcal{F}$ . Thus  $P_k$  is constant on  $M$  by our assumption. Put  $g := \exp(-P_k)$  and  $Q := gP = \exp(\sum_{i=k+1}^s P_i + \text{an element of } \mathfrak{n}^{k+1})$ . Then

$$\begin{aligned} \omega_0 + \epsilon\omega Z &= \text{Ad}(Q^{-1}g)\Phi_*\omega_0 + (L_{g^{-1}} \circ Q)^*\theta \\ &= \text{Ad}(Q^{-1})\Psi_*\omega_0 + Q^*\theta \end{aligned}$$

where  $\Psi_* := \text{Ad}(g)\Phi_*$ . □

Applying Lemma 3 repeatedly, we see that  $\omega$  is cohomologous to a constant cocycle.



Finally we assume  $H^1(\mathcal{F}) = H^1(\mathfrak{n})$  and prove that  $\rho$  is parameter rigid and  $H^0(\mathcal{F}) = H^0(\mathfrak{n})$ . Parameter rigidity of  $\rho$  follows from Proposition 1. Let  $f \in H^0(\mathcal{F})$ . Fix a non-zero  $\phi \in H^1(\mathfrak{n})$  and denote the corresponding leafwise one-form on  $M$  by  $\tilde{\phi}$ . Then  $f\tilde{\phi} \in H^1(\mathcal{F}) = H^1(\mathfrak{n})$ . Thus there exist  $\psi \in H^1(\mathfrak{n})$  and a  $C^\infty$ -function  $g : M \rightarrow \mathbb{R}$  such that  $f\tilde{\phi} - \tilde{\psi} = d_{\mathcal{F}}g$  where  $\tilde{\psi}$  is the leafwise one-form corresponding to  $\psi$ . Choose  $X \in \mathfrak{n}$  satisfying  $\phi(X) \neq 0$ . Let  $x(t)$  be an integral curve of  $\tilde{X}$  where  $\tilde{X}$  is the vector field corresponding to  $X$ . We have

$$f(x(t))\phi(X) - \psi(X) = \tilde{X}_{x(t)}g = \frac{d}{dt}g(x(t)).$$

By integrating over  $[0, T]$ , we get  $T(f(x(0))\phi(X) - \psi(X)) = g(x(T)) - g(x(0))$ . Since  $g$  is bounded,  $f(x(0))\phi(X) - \psi(X)$  must be zero. Then  $f(x(0)) = \psi(X)/\phi(X)$  and  $f$  is constant on  $M$ .

This completes the proof of Theorem 1.  $\square$

#### 4. A construction of parameter rigid actions

Let us now construct real-valued cocycle rigid actions of nilpotent groups. For the structure theory of nilpotent Lie groups, see [2]. A basis  $X_1, \dots, X_n$  of  $\mathfrak{n}$  is called a *strong Malcev basis* if  $\text{span}_{\mathbb{R}}\{X_1, \dots, X_i\}$  is an ideal of  $\mathfrak{n}$  for each  $i$ . If  $\Gamma$  is a lattice in  $N$ , there exists a strong Malcev basis  $X_1, \dots, X_n$  of  $\mathfrak{n}$  such that  $\Gamma = e^{\mathbb{Z}X_1} \dots e^{\mathbb{Z}X_n}$ . Such a basis is called a *strong Malcev basis strongly based on  $\Gamma$* . Let  $\Gamma$  and  $\Lambda$  be lattices in  $N$ .

*Definition 1.*  $\Lambda$  is *Diophantine with respect to  $\Gamma$*  if there exists a strong Malcev basis  $X_1, \dots, X_n$  of  $\mathfrak{n}$  strongly based on  $\Gamma$  and a strong Malcev basis  $Y_1, \dots, Y_n$  of  $\mathfrak{n}$  strongly based on  $\Lambda$  such that  $Y_i = \sum_{j=1}^i a_{ij}X_j$  for every  $1 \leq i \leq n$ , where  $a_{ii}$  is Diophantine.

Let  $\rho$  be the action of  $\Lambda$  on  $\Gamma \backslash N$  by right multiplication. First we will prove the following.

**THEOREM 4.** *If  $\Lambda$  is Diophantine with respect to  $\Gamma$ , then every real-valued  $C^\infty$  cocycle  $c : \Gamma \backslash N \times \Lambda \rightarrow \mathbb{R}$  over  $\rho$  is cohomologous to a constant cocycle.*

*Proof.* Note that  $X_1$  is in the center of  $\mathfrak{n}$ . Let  $\pi : N \rightarrow \bar{N} := e^{\mathbb{R}X_1} \backslash N$  be the projection. Since  $\Gamma \cap e^{\mathbb{R}X_1} = e^{\mathbb{Z}X_1}$  is a cocompact lattice in  $e^{\mathbb{R}X_1}$ ,  $\bar{\Gamma} := \pi(\Gamma) = e^{\mathbb{R}X_1} \backslash \Gamma e^{\mathbb{R}X_1}$  is a cocompact lattice in  $\bar{N}$ . Let  $\bar{\mathfrak{n}} = \mathbb{R}X_1 \backslash \mathfrak{n}$ . Then  $\bar{X}_2, \dots, \bar{X}_n$  is a strong Malcev basis of  $\bar{\mathfrak{n}}$  strongly based on  $\bar{\Gamma}$ .

We will see that the naturally induced map  $\bar{\pi} : \Gamma \backslash N \rightarrow \bar{\Gamma} \backslash \bar{N}$  is a principal  $S^1$ -bundle. Indeed,

$$\Gamma \backslash \Gamma e^{\mathbb{R}X_1} \hookrightarrow \Gamma \backslash N \twoheadrightarrow \Gamma e^{\mathbb{R}X_1} \backslash N$$

is a principal  $\Gamma \backslash \Gamma e^{\mathbb{R}X_1}$ -bundle, and we have

$$\Gamma \backslash \Gamma e^{\mathbb{R}X_1} \simeq \Gamma \cap e^{\mathbb{R}X_1} \backslash e^{\mathbb{R}X_1} = e^{\mathbb{Z}X_1} \backslash e^{\mathbb{R}X_1} \simeq \mathbb{Z} \backslash \mathbb{R}$$

and the following diagram.

$$\begin{array}{ccc} e^{\mathbb{R}X_1} \backslash \Gamma e^{\mathbb{R}X_1} & \hookrightarrow & e^{\mathbb{R}X_1} \backslash N \twoheadrightarrow \Gamma e^{\mathbb{R}X_1} \backslash N \\ & \downarrow & \nearrow \sim \\ & \bar{\Gamma} \backslash \bar{N} & \end{array}$$

Since  $\Lambda \cap e^{\mathbb{R}X_1} = \Lambda \cap e^{\mathbb{R}Y_1} = e^{\mathbb{Z}Y_1}$  is a cocompact lattice in  $e^{\mathbb{R}X_1}$ , then  $\bar{\Lambda} := \pi(\Lambda)$  is a cocompact lattice in  $\bar{N}$ . Furthermore,  $\bar{Y}_2, \dots, \bar{Y}_n$  is a strong Malcev basis of  $\bar{\mathfrak{n}}$  strongly based on  $\bar{\Lambda}$  and  $\bar{Y}_i = \sum_{j=2}^i a_{ij} \bar{X}_j$  where  $a_{ii}$  is Diophantine. Therefore  $\bar{\Lambda}$  is Diophantine with respect to  $\bar{\Gamma}$ .

Since  $\bar{\pi}$  is  $\Lambda$ -equivariant, the action  $\rho$  of  $\Lambda$  when restricted to  $e^{\mathbb{Z}Y_1}$ , preserves fibers of  $\bar{\pi}$ .

Let  $z \in \bar{\Gamma} \backslash \bar{N}$ . Choose a point  $\Gamma x$  in  $\bar{\pi}^{-1}(z)$ . Then we have a trivialization

$$\iota_{\Gamma x} : \mathbb{Z} \backslash \mathbb{R} \simeq \bar{\pi}^{-1}(z)$$

of  $\bar{\pi}^{-1}(z)$  given by  $\iota_{\Gamma x}(s) = \Gamma e^{sX_1}x$ . Note that if we take another point  $\Gamma y \in \bar{\pi}^{-1}(z)$ ,  $\iota_{\Gamma y}^{-1} \circ \iota_{\Gamma x} : \mathbb{Z} \backslash \mathbb{R} \rightarrow \mathbb{Z} \backslash \mathbb{R}$  is a rotation.

Let  $Y_1 = aX_1$ , where  $a$  is Diophantine. If we identify  $\bar{\pi}^{-1}(z)$  with  $\mathbb{Z} \backslash \mathbb{R}$  by  $\iota_{\Gamma x}$ , then the action of  $e^{Y_1}$  on  $\mathbb{Z} \backslash \mathbb{R}$  is  $s \mapsto s + a$ .

Let  $\mu_z$  be the normalized Haar measure naturally defined on  $\bar{\pi}^{-1}(z)$ ,  $\mu$  the  $N$ -invariant probability measure on  $\Gamma \backslash N$  and  $\nu$  the  $\bar{N}$ -invariant probability measure on  $\bar{\Gamma} \backslash \bar{N}$ . For any  $f \in C(\Gamma \backslash N)$ ,

$$\int_{\Gamma \backslash N} f \, d\mu = \int_{\bar{\Gamma} \backslash \bar{N}} \int_{\bar{\pi}^{-1}(z)} f \, d\mu_z \, d\nu. \quad (2)$$

LEMMA 4.  $\rho$  is ergodic with respect to  $\mu$ .

*Proof.* We use induction on  $n$ . If  $n = 1$ ,  $\rho$  is an irrational rotation on  $\mathbb{Z} \backslash \mathbb{R}$ , and hence the result is well known. In general, let  $f : \Gamma \backslash N \rightarrow \mathbb{C}$  be a  $\Lambda$ -invariant  $L^2$ -function with  $\int_{\Gamma \backslash N} f \, d\mu = 0$ . Since the action of  $e^{\mathbb{Z}Y_1}$  on  $\bar{\pi}^{-1}(z)$  is ergodic,  $f|_{\bar{\pi}^{-1}(z)}$  is constant  $\mu_z$ -almost everywhere. We denote this constant by  $g(z)$ . Then  $g : \bar{\Gamma} \backslash \bar{N} \rightarrow \mathbb{C}$  is a  $\bar{\Lambda}$ -invariant measurable function. By induction,  $g$  is constant  $\nu$ -almost everywhere. By (2), this constant must be zero. Therefore  $f$  is zero  $\mu$ -almost everywhere.  $\square$

Let  $c : \Gamma \backslash N \times \Lambda \rightarrow \mathbb{R}$  be a  $C^\infty$ -cocycle over  $\rho$ . We must show that  $c$  is cohomologous to a constant cocycle  $c_0 : \Lambda \rightarrow \mathbb{R}$  where  $c_0(\lambda) := \int_{\Gamma \backslash N} c(x, \lambda) \, d\mu(x)$ . Therefore we may assume that  $\int_{\Gamma \backslash N} c(x, \lambda) \, d\mu(x) = 0$  for all  $\lambda \in \Lambda$ , and we will show that  $c$  is a coboundary. We prove this by induction on  $n$ . When  $n = 1$ ,  $\rho$  is a Diophantine rotation on  $\mathbb{Z} \backslash \mathbb{R}$ , and hence the result is well known.

LEMMA 5. For all  $m \in \mathbb{Z}$ ,

$$\int_{\bar{\pi}^{-1}(z)} c(s, e^{mY_1}) \, d\mu_z(s) = 0.$$

*Proof.* Fix  $m$  and put  $g(z) = \int_{\bar{\pi}^{-1}(z)} c(s, e^{mY_1}) \, d\mu_z(s)$ . For any  $\lambda \in \Lambda$ , the cocycle equation gives  $c(x, \lambda) + c(x\lambda, e^{mY_1}) = c(x, e^{mY_1}) + c(xe^{mY_1}, \lambda)$ . By integrating this equation on  $\bar{\pi}^{-1}(z)$ , we get  $g(z\pi(\lambda)) = g(z)$ . Since the action of  $\bar{\Lambda}$  on  $\bar{\Gamma} \backslash \bar{N}$  is ergodic,  $g$  is constant. By (2),  $g$  must be zero.  $\square$

Let  $f : \mathbb{Z} \backslash \mathbb{R} \xrightarrow{\iota_{\Gamma x}} \bar{\pi}^{-1}(z) \xrightarrow{c(\cdot, e^{Y_1})} \mathbb{R}$ . We define

$$h_z(\iota_{\Gamma x}(s)) = \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{\hat{f}(k)}{-1 + e^{2\pi i k a}} e^{2\pi i k s}.$$

Then  $h_z : \bar{\pi}^{-1}(z) \rightarrow \mathbb{R}$  is  $C^\infty$ , since  $f$  is  $C^\infty$  and  $a$  is Diophantine. By Lemma 5, we have

$$c(\iota_{\Gamma_X}(s), e^{Y_1}) = -h_z(\iota_{\Gamma_X}(s)) + h_z(\iota_{\Gamma_X} e^{Y_1}).$$

If we choose another point  $\Gamma e^{s_0 X_1} x \in \bar{\pi}^{-1}(z)$  to define  $h_z$ , then

$$\begin{aligned} h_z(\iota_{\Gamma_X}(s)) &= h_z(\Gamma e^{s X_1} x) = h_z(\iota_{\Gamma e^{s_0 X_1} x}(s - s_0)) \\ &= \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{-1 + e^{2\pi i k a}} \int_0^1 c(\Gamma e^{(u+s_0)X_1} x, e^{Y_1}) e^{-2\pi i k u} du e^{2\pi i k (s-s_0)} \\ &= \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{-1 + e^{2\pi i k a}} \int_0^1 f(u + s_0) e^{-2\pi i k u} du e^{2\pi i k (s-s_0)} \\ &= \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{\hat{f}(k)}{-1 + e^{2\pi i k a}} e^{2\pi i k s}, \end{aligned}$$

so  $h_z$  is determined only by  $z$ . Define  $h : \Gamma \backslash N \rightarrow \mathbb{R}$  by  $h|_{\bar{\pi}^{-1}(z)} = h_z$ . Then for all  $x \in \Gamma \backslash N$  and  $m \in \mathbb{Z}$ ,  $c(x, e^{m Y_1}) = -h(x) + h(x e^{m Y_1})$ .

Let  $U \subset \bar{\Gamma} \backslash \bar{N}$  be open and  $\sigma : U \rightarrow \bar{\pi}^{-1}(U)$  a section of  $\bar{\pi}$ . Then we have a trivialization  $\mathbb{Z} \backslash \mathbb{R} \times U \simeq \bar{\pi}^{-1}(U)$  which sends  $(s, z)$  to  $\iota_{\sigma(z)}(s) = \Gamma e^{s X_1} \sigma(z)$ . Hence

$$h(\iota_{\sigma(z)}(s)) = \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{-1 + e^{2\pi i k a}} \int_0^1 c(\iota_{\sigma(z)}(u), e^{Y_1}) e^{-2\pi i k u} du e^{2\pi i k s}$$

on  $\bar{\pi}^{-1}(U)$ . The following lemma shows  $h$  is  $C^\infty$  on  $\Gamma \backslash N$ .

LEMMA 6. Let  $U \subset \mathbb{R}^n$  be open and let  $f : \mathbb{Z} \backslash \mathbb{R} \times U \rightarrow \mathbb{R}$  be a  $C^\infty$ -function. Define

$$h(s, z) = \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{-1 + e^{2\pi i k a}} \hat{f}_z(k) e^{2\pi i k s},$$

where  $f_z(u) = f(u, z)$ . Then  $h : \mathbb{Z} \backslash \mathbb{R} \times U \rightarrow \mathbb{R}$  is  $C^\infty$ .

*Proof.* Let  $V$  be open such that  $\bar{V} \subset U$  and  $\bar{V}$  is compact. We will show that  $h$  is  $C^\infty$  on  $\mathbb{Z} \backslash \mathbb{R} \times V$ . Choose constants  $C, \alpha > 0$  such that  $|-1 + e^{2\pi i k a}| \geq C|k|^{-\alpha}$  for all  $k \in \mathbb{Z} \setminus \{0\}$ .

We will first prove that  $h$  is continuous. Since for any  $m \in \mathbb{Z}_{>0}$ ,

$$\frac{\partial^m f_z}{\partial s^m}(s) = \sum_{k \in \mathbb{Z}} (2\pi i k)^m \hat{f}_z(k) e^{2\pi i k s}$$

in  $L^2(\mathbb{Z} \backslash \mathbb{R})$ ,

$$\begin{aligned} \left\| \frac{\partial^m f_z}{\partial s^m} \right\|_2^2 &= \sum_{k \in \mathbb{Z}} |(2\pi i k)^m \hat{f}_z(k)|^2 \\ &\geq (2\pi)^{2m} |k|^{2m} |\hat{f}_z(k)|^2 \geq |k|^{2m} |\hat{f}_z(k)|^2. \end{aligned}$$

Since

$$\left\| \frac{\partial^m f_z}{\partial s^m} \right\|_2 = \left( \int_0^1 \left| \frac{\partial^m}{\partial s^m} f(s, z) \right|^2 ds \right)^{\frac{1}{2}}$$

is continuous in  $z$ , there exists  $M > 0$  such that  $\|\partial^m f_z / \partial s^m\|_2 < M$  for every  $z \in \bar{V}$ . Hence, for all  $k \in \mathbb{Z}$  and  $z \in \bar{V}$ ,  $|k|^m |\widehat{f_z}(k)| \leq M$ . Therefore, for any  $z \in \bar{V}$ ,

$$\begin{aligned} \sum_{k \in \mathbb{Z} \setminus \{0\}} \left| \frac{1}{-1 + e^{2\pi i k a}} \widehat{f_z}(k) e^{2\pi i k s} \right| &\leq C^{-1} \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{|k|^2} |k|^{\alpha+2} |\widehat{f_z}(k)| \\ &\leq C^{-1} M \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{|k|^2} < \infty. \end{aligned}$$

This implies continuity of  $h$  on  $\mathbb{Z} \setminus \mathbb{R} \times \bar{V}$ .

We have

$$\frac{\partial h}{\partial s}(s, z) = \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{2\pi i k}{-1 + e^{2\pi i k a}} \widehat{f_z}(k) e^{2\pi i k s}.$$

Thus a similar argument shows that  $\partial h / \partial s$  is continuous.

Let  $z = (z_1, \dots, z_n)$ . For any  $z \in \bar{V}$ ,

$$\begin{aligned} \left| \frac{\partial}{\partial z_j} \left( \frac{1}{-1 + e^{2\pi i k a}} \widehat{f_z}(k) e^{2\pi i k s} \right) \right| &= \left| \frac{1}{-1 + e^{2\pi i k a}} \widehat{\frac{\partial f}{\partial z_j}(\cdot, z)}(k) e^{2\pi i k s} \right| \\ &\leq C^{-1} \frac{1}{|k|^2} |k|^{\alpha+2} \left| \widehat{\frac{\partial f}{\partial z_j}(\cdot, z)}(k) \right| \\ &\leq C^{-1} M' \frac{1}{|k|^2} \in L^1(\mathbb{Z} \setminus \{0\}). \end{aligned}$$

Thus

$$\frac{\partial h}{\partial z_j}(s, z) = \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{-1 + e^{2\pi i k a}} \widehat{\frac{\partial f}{\partial z_j}(\cdot, z)}(k) e^{2\pi i k s}.$$

Hence  $\partial h / \partial z_j$  is continuous by an argument similar to those above. For higher derivatives of  $h$ , continue this procedure.  $\square$

Set  $c_1(x, \lambda) = c(x, \lambda) + h(x) - h(x\lambda)$ . Then  $c_1 : \Gamma \setminus N \times \Lambda \rightarrow \mathbb{R}$  is a  $C^\infty$ -cocycle and  $c_1(x, e^{mY_1}) = 0$ . Thus for any  $\lambda \in \Lambda$ , the cocycle equation implies  $c_1(x, \lambda) = c_1(xe^{Y_1}, \lambda)$ . Since the action of  $e^{\mathbb{Z}Y_1}$  on  $\bar{\pi}^{-1}(z)$  is ergodic,  $c_1(x, \lambda)$  is constant on  $\bar{\pi}^{-1}(z)$ . Therefore we can define a cocycle  $\bar{c} : \bar{\Gamma} \setminus \bar{N} \times \bar{\Lambda} \rightarrow \mathbb{R}$  by  $\bar{c}(\bar{\pi}(x), \pi(\lambda)) = c_1(x, \lambda)$ . Indeed, if  $\bar{\pi}(x) = \bar{\pi}(y)$  and  $\pi(\lambda) = \pi(\lambda')$ , then there exists an  $m \in \mathbb{Z}$  with  $\lambda = e^{mY_1} \lambda'$ , so that

$$c_1(x, \lambda) = c_1(x, e^{mY_1} \lambda') = c_1(xe^{mY_1}, \lambda') = c_1(y, \lambda').$$

Furthermore,

$$\begin{aligned} \int_{\bar{\Gamma} \setminus \bar{N}} \bar{c}(x, \pi(\lambda)) d\nu(z) &= \int_{\bar{\Gamma} \setminus \bar{N}} \int_{\bar{\pi}^{-1}(z)} c_1(s, \lambda) d\mu_z(s) d\nu(z) \\ &= \int_{\Gamma \setminus N} c_1(x, \lambda) d\mu(x) = 0. \end{aligned}$$

By induction, there exists a  $C^\infty$ -function  $P : \bar{\Gamma} \setminus \bar{N} \rightarrow \mathbb{R}$  such that  $\bar{c}(z, \pi(\lambda)) = -P(z) + P(z\pi(\lambda))$ . Put  $Q = P \circ \bar{\pi}$ . Then  $c_1(x, \lambda) = \bar{c}(\bar{\pi}(x), \pi(\lambda)) = -Q(x) + Q(x\lambda)$ . This proves Theorem 4.  $\square$

*Proof of Theorem 3.* Let  $\tilde{\rho} : M \times N \rightarrow M$  be the suspension of  $\rho : \Gamma \backslash N \times \Lambda \rightarrow \Gamma \backslash N$  where  $M = \Gamma \backslash N \times_{\Lambda} N$  is a compact manifold. Then  $\tilde{\rho}$  is locally free and let  $\mathcal{F}$  be its orbit foliation. We have

$$H^1(\mathcal{F}) \simeq H^1(\Lambda, C^\infty(\Gamma \backslash N))$$

by [6], where the right hand side is the first cohomology of the  $\Lambda$ -module  $C^\infty(\Gamma \backslash N)$  obtained by  $\rho$ . It is easy to prove that  $\text{Hom}(\Lambda, \mathbb{R}) \rightarrow H^1(\Lambda, C^\infty(\Gamma \backslash N))$  is injective. By Theorem 4,

$$H^1(\Lambda, C^\infty(\Gamma \backslash N)) = \text{Hom}(\Lambda, \mathbb{R}).$$

LEMMA 7.  $\dim \text{Hom}(\Lambda, \mathbb{R}) = \dim H^1(\mathfrak{n})$ .

*Proof.* Recall that  $[N, N] \backslash \Lambda[N, N]$  is a cocompact lattice in  $[N, N] \backslash N$  and that  $[\Lambda, \Lambda] \backslash (\Lambda \cap [N, N])$  is finite. Since

$$0 \rightarrow [\Lambda, \Lambda] \backslash (\Lambda \cap [N, N]) \rightarrow [\Lambda, \Lambda] \backslash \Lambda \rightarrow [N, N] \backslash \Lambda[N, N] \rightarrow 0$$

is exact, we have

$$\text{rank } [\Lambda, \Lambda] \backslash \Lambda = \text{rank } [N, N] \backslash \Lambda[N, N] = \dim [N, N] \backslash N.$$

Thus

$$\begin{aligned} \dim \text{Hom}(\Lambda, \mathbb{R}) &= \dim \text{Hom}([\Lambda, \Lambda] \backslash \Lambda, \mathbb{R}) \\ &= \text{rank } [\Lambda, \Lambda] \backslash \Lambda \\ &= \dim [N, N] \backslash N \\ &= \dim \text{Hom}_{\mathbb{R}}([\mathfrak{n}, \mathfrak{n}] \backslash \mathfrak{n}, \mathbb{R}) \\ &= \dim H^1(\mathfrak{n}). \end{aligned} \quad \square$$

Therefore we obtain

$$H^1(\mathcal{F}) = H^1(\mathfrak{n}).$$

This proves Theorem 3. □

### 5. Existence of Diophantine lattices

Let  $\mathfrak{n}_{\mathbb{Q}}$  be a rational structure of  $\mathfrak{n}$ . We construct Diophantine lattices when  $\mathfrak{n}_{\mathbb{Q}}$  admits a graduation. Namely, we assume that  $\mathfrak{n}_{\mathbb{Q}}$  has a sequence  $V_i$  of  $\mathbb{Q}$ -subspaces such that  $\mathfrak{n}_{\mathbb{Q}} = \bigoplus_{i=1}^k V_i$  and  $[V_i, V_j] \subset V_{i+j}$ . Let  $X_1, \dots, X_n$  be a  $\mathbb{Q}$ -basis of  $\mathfrak{n}_{\mathbb{Q}}$  such that  $X_1, \dots, X_{i_1} \in V_k$ ,  $X_{i_1+1}, \dots, X_{i_2} \in V_{k-1}$ ,  $\dots$ ,  $X_{i_{k-1}+1}, \dots, X_n \in V_1$ . Then  $X_1, \dots, X_n$  is a strong Malcev basis of  $\mathfrak{n}$  with rational structure constants. Multiplying  $X_1, \dots, X_n$  by an integer if necessary, we may assume that  $\Gamma := e^{\mathbb{Z}X_1} \dots e^{\mathbb{Z}X_n}$  is a cocompact lattice in  $N$ . Let  $\alpha$  be a root of an irreducible polynomial of degree  $k+1$  over  $\mathbb{Q}$ . Since  $\alpha, \alpha^2, \dots, \alpha^k$  are irrational algebraic numbers, they are Diophantine. If we define a linear map  $\varphi : \mathfrak{n} \rightarrow \mathfrak{n}$  by  $\varphi(X) = \alpha^i X$  for  $X \in V_i \otimes \mathbb{R}$ , then  $\varphi$  is an automorphism of Lie algebra  $\mathfrak{n}$ . Put  $Y_i = \varphi(X_i)$ . Then  $Y_1, \dots, Y_n$  is a strong Malcev basis of  $\mathfrak{n}$  strongly based on  $\Lambda := e^{\mathbb{Z}Y_1} \dots e^{\mathbb{Z}Y_n}$ . Thus  $\Lambda$  is Diophantine with respect to  $\Gamma$ .

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